HISTORIA MATHEMATICA **13** (1986), 168–169

NOTE

Remarks to the Third Entry in Gauss' Diary

KARSTEN JOHNSEN

Mathematisches Seminar der Universität Kiel, Olshausenstrasse, 2300 Kiel, Federal Republic of Germany

A new translation and a new interpretation of the third entry in Gauss' diary are presented. © 1986 Academic Press, Inc.

Eine neue Übersetzung und eine neue Interpretation der dritten Eintragung im Tagebuch von Gauss werden vorgestellt. © 1986 Academic Press, Inc.

On présente une nouvelle traduction et une nouvelle interprétation de la troisième note du carnet de Gauss. © 1986 Academic Press, Inc.

AMS 1980 subject classifications: 01A50, 12-03.
KEY WORDS: Gauss, Gauss' diary, cyclotomic field, cyclotomic polynomial, irreducibility.

On 12 April 1796, C. F. Gauss observes in his diary:

> Formulae pro cosinibus angulorum peripheriae submultiplorum expressionem generaliorem non admittent nisi in duabus periodis. [Gauss 1901, No. 3]

We propose the following translation which differs from [Neumann 1978, 143], [Gauss 1981, 61], and [Gray 1984, 107]:

> The formulae for the cosines of the submultiples of angles of a circumference will admit a more general expression only in terms of both of the two periods.

We suggest the following interpretation of this statement:

Let p be an odd prime, $\alpha_j: = 2\pi j/p$, and $a_j: = \cos \alpha_j$ for $1 \leq j \leq (p-1)/2$ (i.e., *formulae pro cosinibus angulorum peripheriae submultiplorum*); furthermore, let $r: = \cos \alpha_1 + i \sin \alpha_1$ and $d_1: = \Sigma_{(k/p)=1} r^k$, $d_2: = \Sigma_{(k/p)=-1} r^k$ (i.e., *duae periodi*; this follows Gauss' own terminology in [Gauss 1801, art. 343] and corresponds to a passage in Gauss' letter to C. L. Gerling of 6 January 1819; see [Neumann 1978, 142]). The numbers d_1 and d_2 are a basis of the field $K: = \mathbb{Q}(d_1)$. Now Gauss' assertion is:

$$\text{If } k_1 a_1 + k_2 a_2 + \cdots + k_{(p-1)/2} a_{(p-1)/2} = 0$$

where $k_1, k_2, \ldots, k_{(p-1)/2} \in K$ is a nontrivial equation (i.e., *expressio generalior*), then there exists a subscript j_0 such that $k_{j_0} \notin \mathbb{R}$ (i.e., *expressionem generaliorem non admittent nisi in duabus periodis*).

This means that $a_1, a_2, \ldots, a_{(p-1)/2}$ are linearly independent over $\mathbb{Q}$, but may be linearly dependent over K. Indeed they are linearly dependent if and only if $p \equiv$

0315-0860/86 $3.00

1 mod 4. For this is equivalent to $K \subseteq \mathbb{Q}(a_1)$ implying $\mathbb{Q}(a_1)$: $K = (p - 1)/4$. But if $p \equiv 3 \bmod 4$ then $\mathbb{Q}(a_1) \cap K = \mathbb{Q}$ and $\mathbb{Q}(a_1)$: $\mathbb{Q} = \mathbb{Q}(r)$: $K = (p - 1)/2$. In this case they are still linearly independent over K. The linear independence of these numbers over $\mathbb{Q}$ yields the linear independence of the pth primitive roots of unity and is therefore equivalent to the irreducibility of the pth cyclotomic polynomial [Gauss 1901, No. 40; Gauss 1801, art. 341].

It does not seem to make sense to read the third entry as a statement on the nth cyclotomic field for arbitrary n. The remark in [Gauss 1981, 88] and [Neumann 1978, 143] on the linear independence of the numbers cos $2\pi j/n$ for $(j, n) = 1$ and $1 \leq j \leq (n - 1)/2$ is not true in general (e.g., it is not true for $n = 4$).

REFERENCES

Gauss, C. F. 1801. *Disquisitiones arithmeticae*. Leipzig: Fleischer. Reprinted in [Gauss 1863, I].

——— 1863. Werke I–XII. Published by the Königliche Gesellschaft der Wissenschaften zu Göttingen, 1863–1933.

——— 1901. Tagebuch (Notizenjournal). First published by F. Klein in *Festschrift zur Feier des hundertjährigen Bestehens der Königlichen Gesellschaft der Wissenschaften zu Göttingen*. Reprinted in [Gauss, 1863, X/I, 483–574].

——— 1981. Mathematisches Tagebuch, 1796–1814. *Ostwalds Klassiker der exakten Wissenschaften, Nr. 256,* 3rd ed. Leipzig.

Gray, J. J. 1984. A commentary on Gauss's diary, 1796–1814, with an English translation. *Expositiones Mathematicae* **2,** 97–130.

Neumann, O. 1978. Über einige Tagebuchnotizen von C. F. Gauss (Zur Entstehung der Kreisteilungstheorie). In Abhandlungen der Akademie der Wissenschaften der DDR, Abteilung Mathematik, Naturwissenschaften und Technik, No. 3. Berlin: Akademie-Verlag.